A Heat Transfer Textbook

Fifth Edition

Solutions Manual for Chapter 1

by

John H. Lienhard IV

and

John H. Lienhard V

Phlogiston Press
Cambridge Massachusetts

Professor John H. Lienhard IV
Department of Mechanical Engineering
University of Houston
4800 Calhoun Road
Houston TX 77204-4792 U.S.A.

Professor John H. Lienhard V
Department of Mechanical Engineering
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge MA 02139-4307 U.S.A.

Names: Lienhard, John H., IV, 1930– | Lienhard, John H., V, 1961–.

Title: A Heat Transfer Textbook: Solutions Manual for Chapter 1 / by John H. Lienhard, IV, and John H. Lienhard, V.

Description: Fifth edition | Cambridge, Massachusetts : Phlogiston Press, 2023 | Includes bibliographical references and index.

Subjects: Heat—Transmission | Mass Transfer.

Published by Phlogiston Press
Cambridge, Massachusetts, U.S.A.

For updates and information, visit:

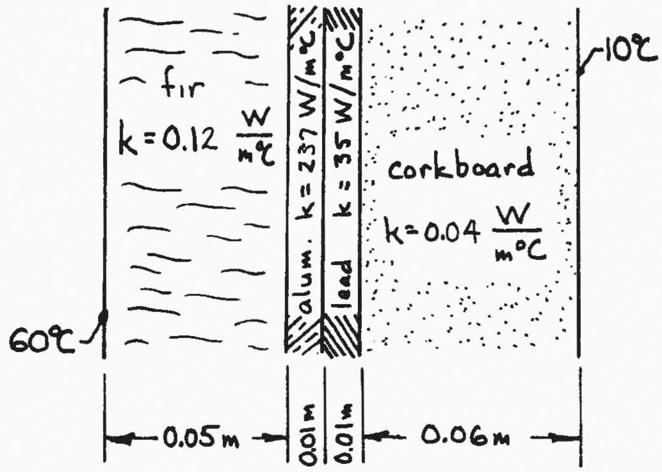
http://ahtt.mit.edu

This copy is:

`Version 1.01 dated 24 February 2023`

1.1 <u>Given</u> the wall shown:

<u>**Find:**</u> The temperature distribution in the wall, and suggest any assumptions that might be made to simplify analysis of the wall



With reference to Example 1.2 we write:

$$\Delta T_{fir}\left(1+\underbrace{\frac{(k/L)_{fir}}{(k/L)_{al}}}_{.00010}+\underbrace{\frac{(k/L)_{fir}}{(k/L)_{lead}}}_{.00069}+\underbrace{\frac{(k/L)_{fir}}{(k/L)_{cbbd}}}_{3.60}\right)$$

$$=(60-10)^\circ C$$

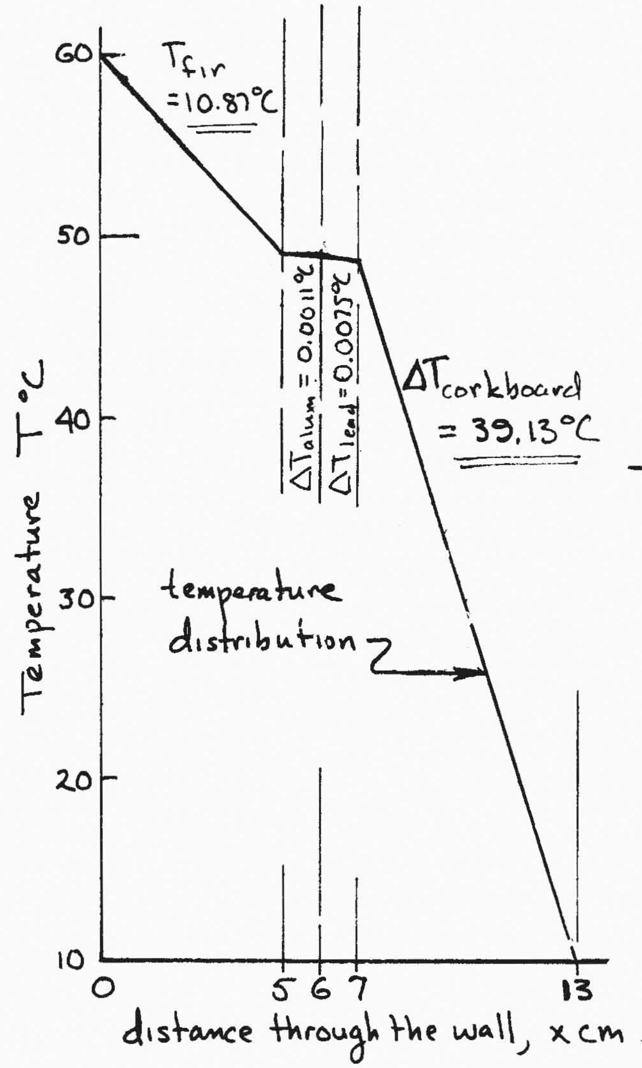
or

$$\Delta T_{fir}=\frac{50}{4.601}=\underline{10.87^\circ C}$$

and the other ΔT's can be obtained from:

$$\Delta T_{other}=\Delta T_{fir}\,\frac{(k/L)_{fir}}{(k/L)_{other}}$$

(values are noted on the graph.)



The high conductivity metal sections offer so little resistance that they can be considered <u>isothermal</u>

1.2 Verify that Newton's Law of cooling, $-\frac{dT_{body}}{dt} \sim (T_{body} - T_\infty)$, is equivalent to $Q \sim (T_{body} - T_\infty)$.

We know that: $$Q = \frac{dU_{body}}{dt} = \frac{d[\rho c(\text{volume of body})(T_{body} - T_{ref})]}{dt}$$
$$= \rho c V \frac{dT_{body}}{dt}$$

$$-\frac{dT_{body}}{dt} = \frac{Q}{\rho c V} \sim \frac{(T_{body} - T_\infty)}{\rho c V}$$

Thus the relations are equivalent if $\rho c V$ is constant.

1.3 5000 W/m^2 are transfered through a 1 cm slab whose cold side is held at -40°C. Find ΔT for each of the 7 materials tabled below. Discuss the results.

$q = k \frac{\Delta T}{L}$,where I is the thickness of the slab.

so

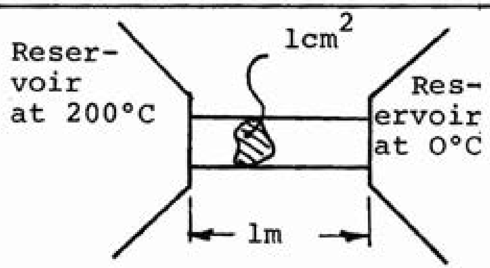
$$\Delta T = (T_{hot} + 40) = \frac{qL}{k} = \frac{(5000\ W/m^2)(0.01m)}{k\ W/m\ °C} = \underline{\frac{50}{k}\ °C}$$

Material	$k\ \frac{W}{m°C}$	$\Delta T = \frac{50}{k}°C$	$T_{hot} = (\Delta T - 40)°C$	Comments
Silver	429	0.12	-39.88	Very small ΔT
Aluminum	239	0.21	-39.79	Very small ΔT
Mild steel	57	0.88	-39.12	Pretty small ΔT
Ice	2.215	22.57	-17.43	Larger ΔT, but not enough to melt the ice.
Spruce	0.11	455	415	The wood would char and burn the hot side. k will deviate from the given value.
85% mag. insul.	~0.06	833	793	Very large ΔT. The correct effective value of k will differ greatly from 0.06.
Silica aerogel	0.024	2083	2043	Enormous value of ΔT. The insulation will be ruined and ΔT will drop to a lower value.

1.4 Explain why the heat diffusion equation, $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$, shows that in transient conduction T depends on $\alpha = k/\rho c$, but solutions of steady conduction problems involve only k.

In a transient problem (one in which the local temperature of a body changes) some of the heat flow is either stored in, of drawn from, the body. Consequently, both k (which determines the through-flow of heat in accordance with Fourier's Law) and the volumetric heat capacity, ρc, (which determines the storage), appear in the heat diffusion equation. They happen to appear in the ratio, α. During steady conduction, the local temperature never changes so ρc is not involved in the problem.

1.5 Consider the copper ($k = 391$ W/m°C) rod connecting two thermal reservoirs as shown: The system is steady.



__Find:__ Rate of change of S for

a) the first reservoir, c) the rod, and
b) the second reservoir, d) the universe,

as a result of the process. Relate the results to the Second Law of Thermodynamics.

First, establish Q: $Q = kA\frac{\Delta T}{L} = 391\ \frac{W}{m°C}\ (0.01m)^2\ \frac{(200-0)°C}{1m} = \underline{7.82W}$

Then: $\dot{S}_{res\#1} = \frac{-Q}{T_1} = -\frac{7.82}{200+273} = \underline{\underline{-0.0165\ \frac{W}{°K}}}$ ← a

$\dot{S}_{res\#2} = \frac{+Q}{T_2} = \frac{7.82}{273} = \underline{\underline{0.0286\ \frac{W}{°K}}}$ ← b

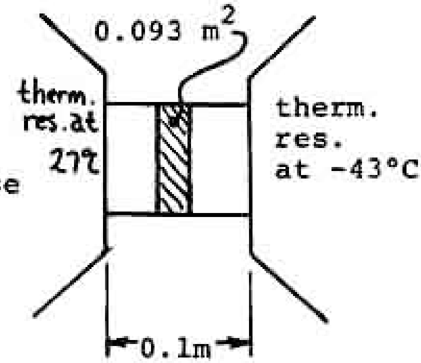
$\dot{S}_{rod} = \underline{0}$ since it is at steady state. ← c

$\dot{S}_{un} = \dot{S}_{res\#1} + \dot{S}_{res\#2} + \dot{S}_{rod} = \underline{\underline{+0.0121\ \frac{W}{°K}}}$ ← d

The rate of change of entropy of the universe is positive as a result of the process, as the Second Law of Thermodynamics requires it to be.

1.6 Heat is conducted steadily through a wall with k = 0.14 W/m²C as shown.

Find: The rates of change of entropy of the components and the universe



This problem is exactly the same as 1.5 except the heat transfer rate is higher:

$$Q = 0.14(.093)\,\frac{70}{0.1} = \underline{9.114\ \mathrm{W}}$$

so

$$\dot S_{res.\#1} = \underline{\underline{-0.03038}},\quad \dot S_{res.\#2} = \underline{\underline{+0.03963}};\quad \dot S_{rod} = \underline{\underline{0}};$$

$$\underline{\underline{\dot S_{un} = 0.00925\ \frac{\mathrm{W}}{°\mathrm{K}}}}$$

1.7 Replace the thermal reservoirs in 1.6 with adiabatic walls. If ρc = 1133.4 kJ/m³C,

Find: a) the final equilibrium temperature, T_f, of the slab;

b) ΔS for the process; c) Is the Second Law satisfied?

a.) The final temperature is obviously just a simple mean:

$$T_f = \frac{27+(-43)}{2} = \underline{\underline{-8°C}}$$

b.) For a solid, the specific entropy is $S-S_{ref} = c\,\ell n\, T/T_{ref}$

Thus we may write at any section of the slab:

$$S_{final}-S_{initial} = c\,\ell n\, T_f/T_i$$
$$= -c\,\ell n\, T_i/T_f$$

Thus: $\Delta S = \rho A \int_{x=0}^{x=0.1m} (S_f-S_i)dx = -\rho c A \int_0^{0.1} (\ell n\, T_i/T_f)dx$

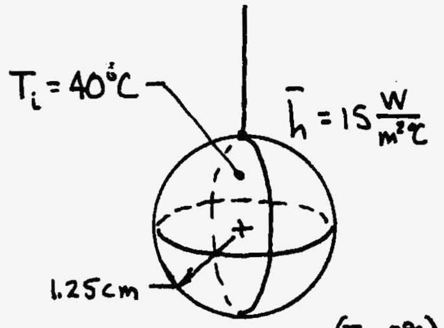
However the initial linear temperature profile is T_i=300°K-(700 °K/m)x; thus dx = $-dT_i/700$ and

$$\Delta S = \frac{\rho c A}{700}\int_{300}^{230} (\ell n\, T_i/T_f)dT_i = 0.1506\left[230\ \ell n\ \frac{230}{273-8} - 230 - 300\ell n\ \frac{300}{265} + 300\right]$$

$$= \underline{\underline{0.03081\ \frac{\mathrm{kJ}}{°\mathrm{K}}}}$$

c.) The net change entropy in this spontaneous, irreversible process is positive as the second Law requires.

1.8 Consider a copper sphere cooling as shown:



Plot: T_{sphere} as a function of time:

The temperature response equation is worked out in Section 1.3. It is:

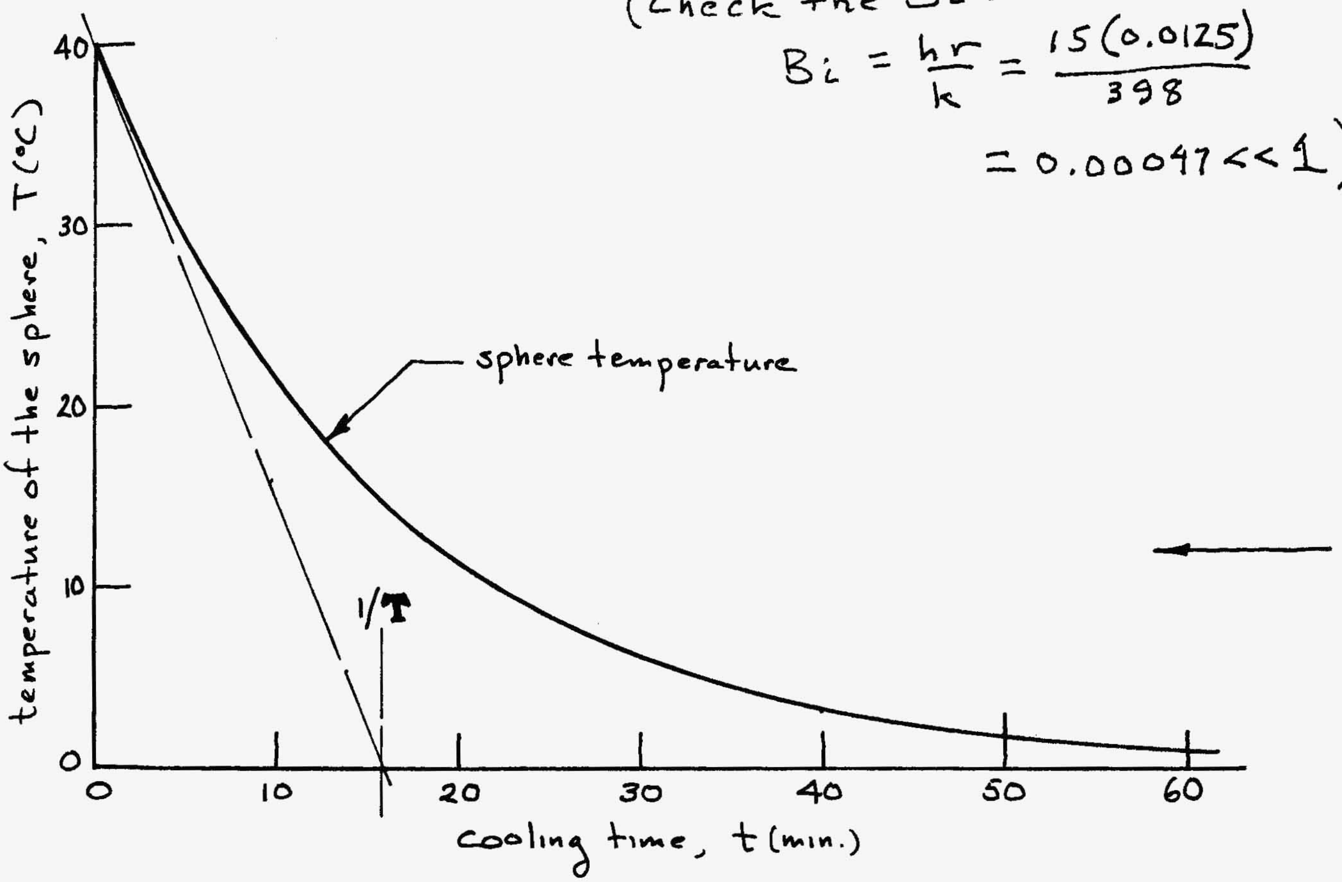
$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{t}{T}\right) = \exp\left[-\frac{6\bar{h}t}{\rho c D}\right]$$

$Bi \ll 1$

so: $$T = \frac{\rho c D}{6\bar{h}} = \frac{m^2 °C}{6(15)\,W} \cdot \frac{8954\,kg}{m^3} \cdot \frac{384\,J}{kg\,°C} \cdot (0.025\,m) \cdot \left(\frac{W \cdot s}{J}\right) = \underline{955\,sec}$$
$$= \underline{15.92\,min.}$$

Thus: $$\underline{\underline{T = 40 \exp\left(-\frac{t\,min}{15.92\,min}\right)}}$$

(Check the Bi:
$$Bi = \frac{hr}{k} = \frac{15(0.0125)}{398}$$
$$= 0.00047 << 1)$$



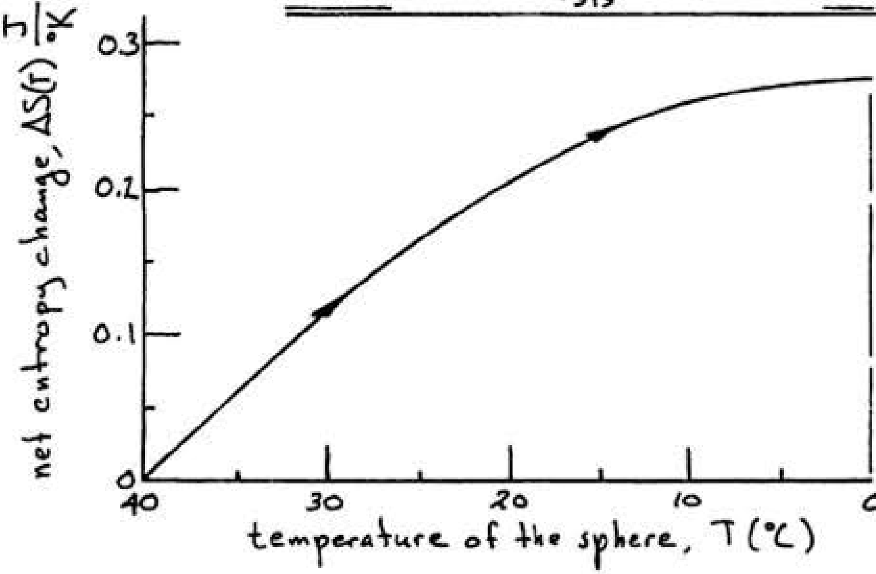
1.9 Determine the total heat transfer in Problem 1.8. Plot the net entropy generated as a function of temperature.

$$Q_{total} = \rho c V \Delta T = 8954(384)\tfrac{4}{3}\pi(0.0125)^3(40) = \underline{\underline{1125\ \text{J}}}$$

$$\frac{dS}{dt} = Q\left[\frac{1}{T_\infty} - \frac{1}{T}\right] = -\rho c V \frac{dT}{dt}\left[\frac{1}{273} - \frac{1}{T}\right] = -\frac{1125}{40}\frac{dT}{dt}\left[\frac{1}{273} - \frac{1}{T}\right]$$

Integrate this result from T = 313°K to T of interest

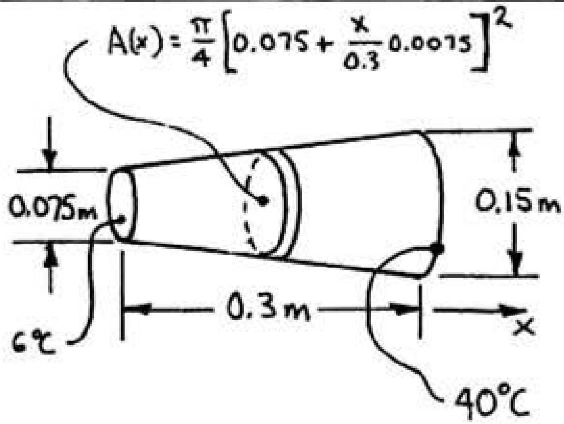
$$\underline{\underline{\Delta S(T) = -28.13\int_{313}^{T}\left[\frac{1}{273} - \frac{1}{T}\right]dT = -28.13\left[\frac{T-313}{273} - \ln\frac{T}{313}\right]\ \frac{\text{J}}{\text{°K}}}}$$



T°K	ΔT°C	ΔS J/°C
313	40	0
303	30	0.117
293	20	0.203
283	10	0.25
273	0	0.27

1.10 Consider the heat transfer through the block shown using k = 0.7 W/m-°C.

Find the net heat transfer, Q



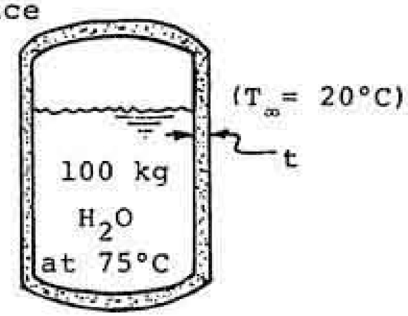
$$Q = \text{constant} = -A(x)\,k\,\frac{dT}{dx}$$

$$\underbrace{-\int_6^{40} dT}_{-34°C} = \frac{4Q}{\pi k}\int_0^{0.3}\frac{dx}{[.075+.25x]^2} = \frac{4Q}{\pi k}\left[\frac{-1}{0.25[.075+.25x]}\right]_0^{0.3}$$

$$\therefore\quad Q = \frac{34°C(\pi)(0.7\ \text{W/m°C})}{4\left[\frac{1}{.25(.075+.075)} - \frac{1}{.25(.075)}\right]} = \underline{\underline{-0.70\ \text{W}}}$$

The minus sign means that the direction of Q is opposite the x-axis.

1.11 The water heater shown, has an external surface area of $1.3m^2$. Select an insulating material and specify its thickness, to keep the water from cooling at more than 3°C/hr. Justify neglecting the thermal resistance of the steel casing and the convective layer.



First determine the heat flux: $q = \frac{Q}{A} = \frac{(\text{mass})c}{A}\frac{dT}{dt} = \frac{(100)(4190)3}{1.3(3600)}$

$$= 268.6 \text{ W/m}^2$$

Then $q = k\frac{\Delta T}{t}$ or $\frac{k}{t} = \frac{268.6}{75-20} = 4.883 \frac{\text{W/m°C}}{\text{m}}$

Now we look at various insulating materials, and see how large t must be

for 85% magnesia: $k = 0.068$, $t_{min} = .014$ m

for 980 kg/m^3 asbestos: $k = 0.14$, $t_{min} =$

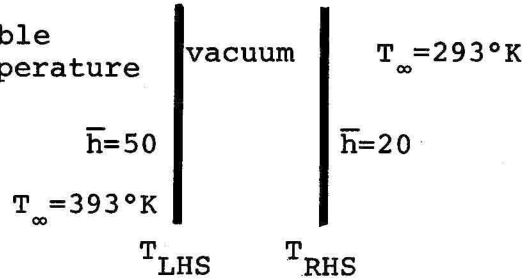
for glass wool: $k = 0.04$, $t_{min} = .0082$ m

Let's specify <u>1-1/2 cm of glass wool</u>. That will be plenty safe

With q = only 269 W/m^2, the temperature drop through a typical $\bar{h} = 10/m^2$-°C, $\Delta T = q/\bar{h} = 27°C$. It appears that neglecting $\bar{h}$ is not justified. We don't really need so much insulation if the air is still around the heater.

And with $q = 269$ W/m^2, temperature drop through, say, 5 mm of steel is $\Delta T = (q/k)(.005 \text{ m}) = \frac{269}{54}(.005) = 0.025°C$, which is entirely negligible.

1.12 The two walls shown offer negligible thermal resistance. Find the temperature on the L.H.S.



$$q = h_{LHS}(373-T_{LHS}) = \sigma(T_{LHS}^4-T_{RHS}^4)$$

$$= h_{RHS}(T_{RHS}-293)$$

To simplify computation, define $t \equiv T/100$. Then

$$50(100)\ (3.73-t_L) = 5.67(t_L^4-t_R^4) = 20(100)\ (t_R-2.93)$$

or

$$3.73-t_L = \frac{5.67}{5000}\ (t_L^4-t_R^4) = 0.4t_R-1.172$$

Then

$$t_L = 4.902-0.4t_R \quad \text{and} \quad t_R = 12.255-2.5t_L$$

and

$$\underline{t_L = 3.73-0.001134\ (t_L^4-[12.255-2.5t_L]^4)}$$

which must be solved by trial and error

t_L	RHS of eqn.
3.4	3.804
3.5	3.731
3.6	3.667
3.65	3.638
3.64	3.643

close enough, T_L = 364°K = $\underline{\underline{91°C}}$

and T_R = $\underline{\underline{42.5°C}}$

1.13

Develop conversion factors for α, g, ρ, σ, F_{1-2}, $\hat{S}$, c:

$$\alpha: \quad \underline{\underline{1\ \frac{m^2}{s}}} = 1\ \frac{m^2}{s}\left(\frac{ft}{.3048\ m}\right)^2 = \underline{\underline{10.764\ \frac{ft^2}{s}}}$$

$$q: \quad \underline{\underline{1\ \frac{W}{m^2}}} = 1\ \frac{J}{m^2\text{-}s}\left(\frac{.3048\ m}{ft}\right)^2\left(\frac{0.00094783\ Btu}{J}\right)\frac{3600s}{hr} = \underline{\underline{0.317\ \frac{Btu}{ft^2\text{-}hr}}}$$

$$\rho: \quad \underline{\underline{1\ \frac{kg}{m^3}}} = 1\ \frac{kg}{m^3}\left(\frac{2.2046\ lbm}{kg}\right)\left(\frac{.3048\ m}{ft}\right)^3 = \underline{\underline{0.06243\ \frac{lbm}{ft^3}}}$$

$$\sigma: \quad \underline{\underline{1\ \frac{W}{m^2\,°K^4}}} = 1\ \frac{W}{m^2\,°K}\ \frac{.317 Btu/ft^2hr}{W/m^2}\left(\frac{°K}{1.8°R}\right)^4 = \underline{\underline{0.03020\ \frac{Btu}{ft^2\text{-}hr°R^4}}}$$

F_{1-2}: 1 = 1 (since F_{1-2} is dimensionless, the conversion factor is <u>unity</u>.)

$$\hat{S}: \quad \underline{\underline{1\ \frac{J}{kg\ mole°K}}} = 1\ \frac{J}{kg\ mole°K}\left(\frac{.0094783 Btu}{J}\right)\left(\frac{kg\ mole}{2.204621 lbm\ mole}\right)$$

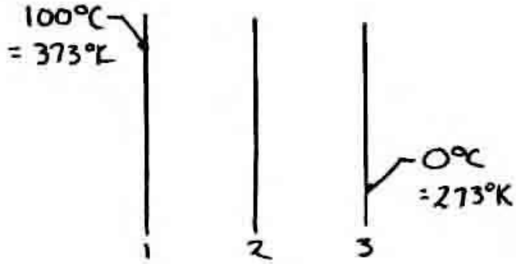
$$\times\left(\frac{°K}{1.8°R}\right)$$

$$= \underline{\underline{0.00023885\ \frac{Btu}{lb_m mole°R}}}$$

$$c: \quad \underline{\underline{1\ \frac{kJ}{kg°C}}} = 1\ \frac{kJ}{kg°C}\left(\frac{0.94783 Btu}{kJ}\right)\left(\frac{kg}{2.204621 lb_m}\right)\left(\frac{°C}{1.8°F}\right) = \underline{\underline{0.23885\frac{Btu}{lb_m\text{-}°F}}}$$

1.14 <u>Given</u> 3 infinite parallel black plates as shown

<u>Req'd.</u> find T_2 if heat is transferred by radiation and the plates are opaque



$$q = \sigma(T_1^4 - T_2^4) = \sigma(T_2^4 - T_3^4) \quad \text{so} \quad 2T_2^4 = T_1^4 + T_3^4$$

$$T_2 = \sqrt[4]{0.5[(373)^4 + (273)^4]} = 334.07^\circ K = \underline{\underline{61.07^\circ C}} \longleftarrow T_2$$

1.15 Same as 1.15 but now there are <u>four</u> plates and we want <u>two</u> intermediate temperatures:

$$q = \underbrace{\sigma(T_1^4 - T_2^4) = \sigma(T_2^4 - T_3^4)} = \sigma(T_3^4 - T_4^4)$$

so :

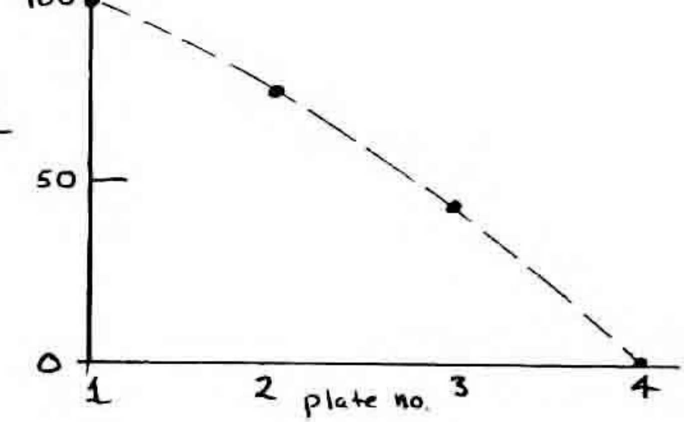
$$T_2 = \underbrace{\sqrt[4]{0.5(T_1^4 + T_3^4)} = \sqrt[4]{2T_3^4 - T_4^4}}$$

$$\frac{3}{2}T_3^4 = \frac{1}{2}T_1^4 + T_4^4 = 1.5233 \times 10^{10} \,{}^\circ K^4$$

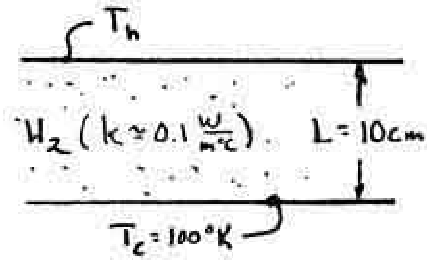
$$\text{so} \quad T_3 = 317.95^\circ K = \underline{\underline{44.95^\circ C}} \longleftarrow T_3$$

Then $\quad T_1^4 - T_2^4 = T_3^4 - T_4^4$

or $\quad T_2 = \sqrt[4]{T_1^4 + T_4^4 - T_3^4} = 348.53^\circ K = \underline{\underline{75.53^\circ C}} \longleftarrow T_2$



1.16 Consider the conduction-radiation configuration shown:



Write the relation:

$$\frac{q_{rad}}{q_{cond.}} = f(N, \Theta \equiv \frac{T_h}{T_c})$$

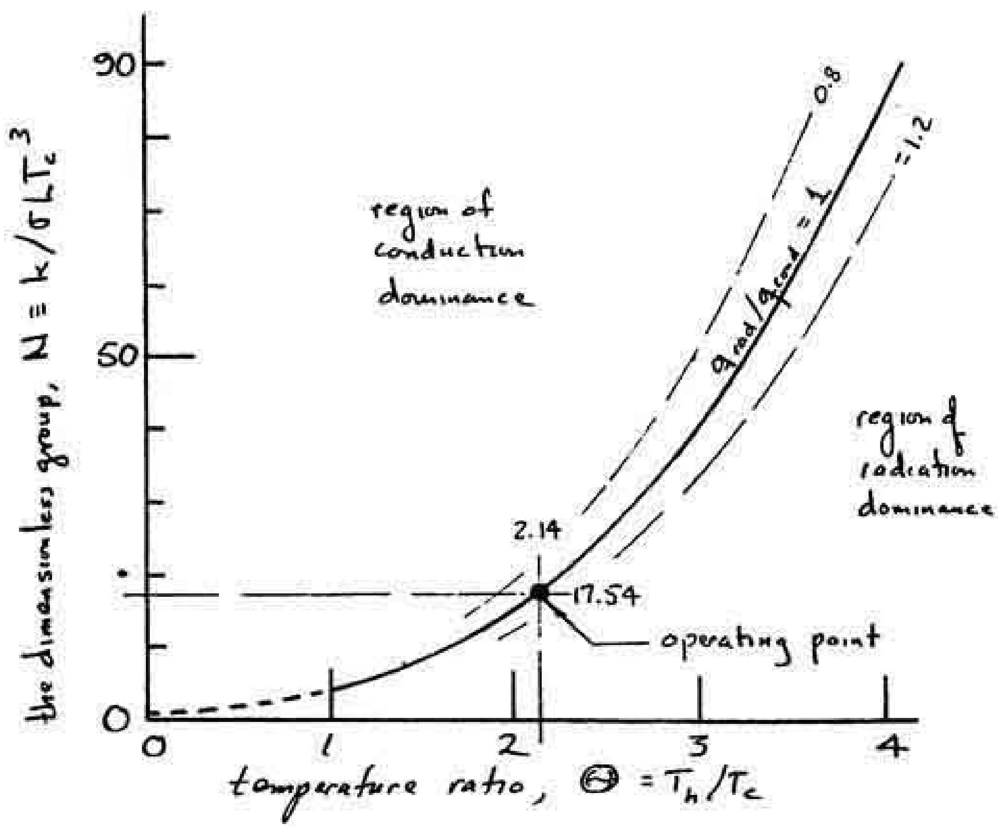
a dimensionless group to be determined

and plot it for $q_{rad}/q_{cond.}$ = 0.8, 1, and 1.2. Identify the given operating point and find T_h.

solution.

$$\frac{q_{rad}}{q_{cond}} = \frac{\sigma(T_h^4 - T_c^4)}{(k/L)(T_h - T_c)} = \frac{\sigma L T_c^3}{k}\frac{(T_h/T_c)^4 - 1}{(T_h/T_c) - 1} = \frac{1}{N}\frac{\Theta^4 - 1}{\Theta - 1}$$

If $q_{rad}/q_{cond} = 1$, then $N = (\Theta^4 - 1)/(\Theta - 1)$ which is plotted below:



In the present problem: $N = \dfrac{0.1}{5.67 \times 10^{-8}(0.1)(100)^3} = \underline{17.54}$

so Θ = 2.14. Therefore the operating temperature is

$$\underline{\underline{T_h = 214°K}}$$

1.17) *Plot $T_{sphere}(t)$ for the sphere shown:*

black sphere initially at 200°C, copper, 1 cm, $k = 390\ W/m\cdot K$; black surroundings at 20°C

$$\rho c V \frac{d(T-T_\infty)}{dt} = -\sigma A (T^4 - T_\infty^4)$$

where $V = \frac{4}{3}\pi R^3$ and $A = 4\pi R^2$. This lumped capacity equation can be used if $hR/2k \ll 1$ where:

$$h = \frac{\sigma A(T_{max}^4 - T_\infty^4)}{T_{max} - T_\infty} = \sigma A(T^2 + T_\infty^2)(T + T_\infty)$$

so:

$$5.67(10^{-8})4\pi(0.01)^2(473^2 + 293^2)(473+293)\frac{0.01}{2(390)}$$

$$= 2.17 \times 10^{-7} \text{ which is very small}$$

Thus:

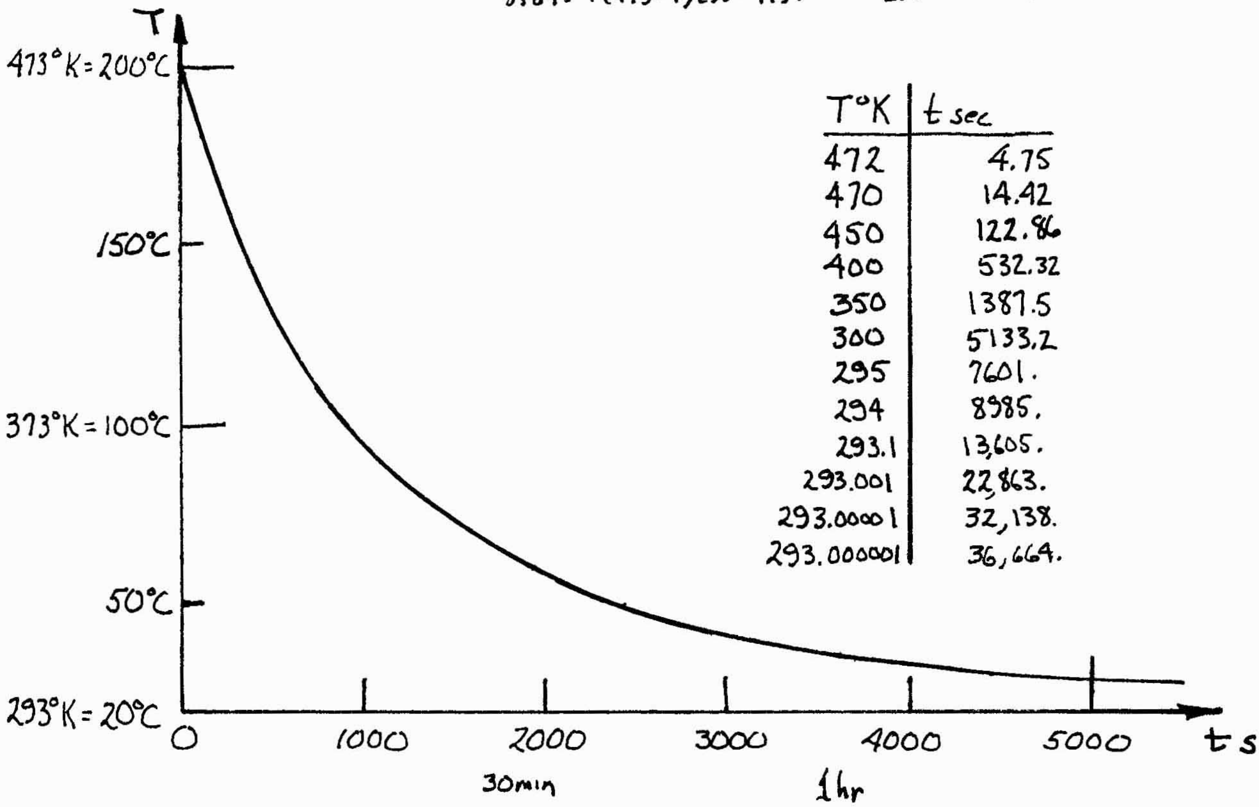
$$\int_{T_i}^{T} \frac{dT}{T^4 - T_\infty^4} = -\int_0^t \frac{3\sigma}{\rho c R}\, dt$$

$$-\frac{1}{2T_\infty^3}\left[\frac{1}{2}\ln\frac{T_\infty + T}{T_\infty - T} + \tan^{-1}\frac{T}{T_\infty}\right]_{T_i}^{T} = -\frac{3\sigma}{\rho c R}t$$

or:

$$t = \frac{8954(389)(0.01)}{6(293)^3 5.67\times10^{-8}}\left[\frac{1}{2}\ln\frac{T_\infty^2 - (T_i - T)T_\infty - TT_i}{T_\infty^2 + (T_i - T)T_\infty - TT_i} + \tan^{-1}\frac{T}{T_\infty} - \tan^{-1}\frac{T_i}{T_\infty}\right]$$

$$t = 4021.2\left[\frac{1}{2}\ln\frac{85849 - (473-T)293 - 473T}{85849 + (473-T)293 - 473T} + \tan^{-1}\frac{T}{293} - 1.0162\right]$$



T°K	t sec
472	4.75
470	14.42
450	122.86
400	532.32
350	1387.5
300	5133.2
295	7601.
294	8985.
293.1	13,605.
293.001	22,863.
293.0001	32,138.
293.000001	36,664.

Problem 1.18 A small instrument package is released from a space vehicle. We can approximate it as a solid aluminum sphere, 4 cm in diameter. The sphere is initially at 303 K and it contains a pressurized hydrogen component that will condense and malfunction at 30 K. If we approximate outer space to be at 0 K, how long will the instrumentation package function properly? Is it legitimate to use the lumped-capacity method?

Solution

$$\rho c V \frac{dT}{dt} = -\sigma A T^4$$

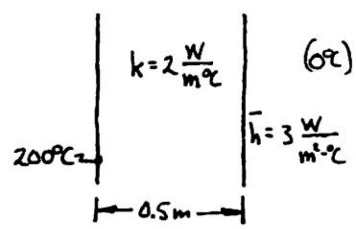
so $$\int_{303}^{30} \frac{dT}{T^4} = \int_0^t \frac{\sigma A}{\rho c V} dt \quad \text{or} \quad \left. \frac{1}{3T^3} \right]_{303}^{30} = \frac{3\sigma}{\rho c R} t$$

Thus:

$$t = \frac{\rho c R}{9\sigma}\left[\frac{1}{30^3} - \frac{1}{303^3}\right] = \frac{2707(905)(0.02)}{9(5.67)10^{-8}}\left[0.000037\right] = 3.55 \times 10^6 \text{ s}$$
$$= 987 \text{ hr}$$
$$= \underline{\underline{41 \text{ days}}}$$

Lumped capacity check (see 1.17) $$Bi = \frac{\sigma T_i^3}{k/L} = \frac{5.67(3.03)^3 10^{-8+6}}{(\sim 250)/0.02}$$
$$= 0.000126 \ll 1 \quad (\text{no problem})$$

1.19 Find the heat flux through the wall shown, and the temperature of the RHS.



$$q = 2\,\frac{200 - T_{RHS}}{0.5} = 3(T_{RHS} - 0)$$

so $\underline{\underline{T_{RHS} = 114.3\,°C}}$

$$\underline{\underline{q = 3(114.3 - 0) = 342.9 \frac{W}{m^2}}}$$

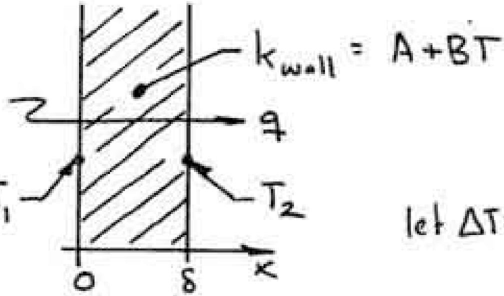
1.20 Prove that T is linear in x during steady planar heat conduction.

$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow \frac{d^2T}{dx^2} = 0$ in this case. Integrate twice and get $\underline{T = C_1 + C_2 x}$

Now at $x=0$, $T = T_{lhs}$, so $\underline{C_1 = T_{lhs}}$ and at $x = L$, $T = T_{rhs}$ so $C_2 = (T_{rhs} - T_{lhs})/L$

It follows that: $$\underline{\underline{T = T_{rhs} + \frac{T_{rhs} - T_{lhs}}{L} x}}$$

1.21 Consider the wall shown.



let $\Delta T \equiv T_1 - T_2$

Find an equation for q

Fourier's Law: $q = -(A+BT)\frac{dT}{dx}$ where q = constant

therefore
$$\int_0^\delta q\,dx = -\int_{T_1}^{T_2}(A+BT)\,dT$$

$$q\delta = -A(T_2-T_1) - \frac{B}{2}(T_2^2-T_1^2)$$

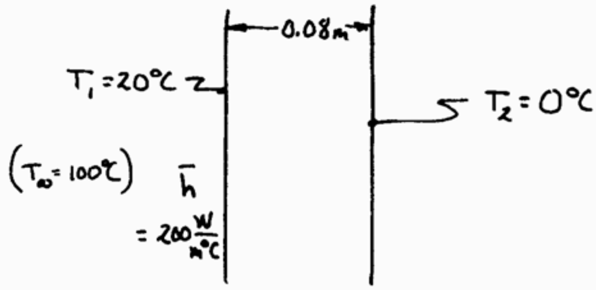
$$\underline{\underline{q = \frac{A\Delta T}{\delta} + \frac{B(T_2+T_1)}{2\delta}\Delta T}}$$

There is a very important ramification of this result: by rearranging it we obtain

$$q = \frac{\left(A + B\frac{T_2+T_1}{2}\right)\Delta T}{\delta} = \frac{k(T_{avg})\,\Delta T}{\delta}$$

This means that q may be evaluated precisely in plane steady heat transfer if we use the mean value of k.

1.22 Find k for the wall shown: what might it be made of?



q through wall

$= q$ through the layer on the left

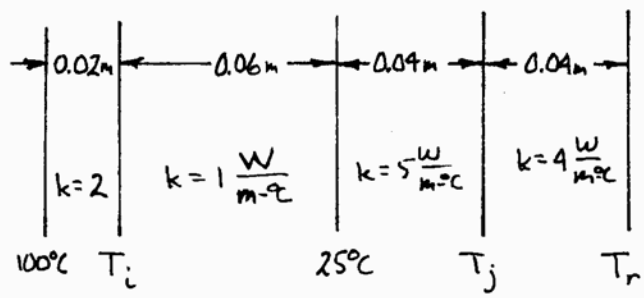
$= \bar{h}(T_\infty - T_1) = 200(100-20) = 16000 \text{ W/m}^2$

but q also equals $k\frac{\Delta T}{L}$: $16000 = k\frac{(20-0)}{0.08}$

so: $\underline{\underline{k = 64 \frac{W}{m°C}}}$

From Table A.1 we find nothing at $T = \frac{20+0}{2} = 10°C$ with exactly this k, but it is very close to steel, tin, or platinum. It might be slightly impure iron, tin, or platinum; or perhaps an alloy -- maybe basically nickel.

1.23 Find T_i and T_j for the wall shown



$$q = k\frac{\Delta T}{L} = \underbrace{\frac{2(100-T_i)}{0.02} = \frac{1(T_i - 25)}{0.06}}$$

$600 - 6T_i - T_i + 25 = 0 \qquad$ so $T_i = \frac{625}{7} = \underline{\underline{89.3°C}}$

$$q = \frac{2(100-89.3)}{0.02} = \underbrace{1070 \frac{W}{m^2} = \frac{5(25-T_j)}{0.04}}$$

$T_j = -0.008(1070) + 25 = \underline{\underline{16.44°C}}$

$q = 1070 = \frac{4(16.44 - T_r)}{0.04} = 1644 - 100T_r \;; \qquad T_r = \underline{\underline{5.74°C}}$

1.24 An aluminum beverage can (12 cm high and 6 cm in diameter) is initially at refrigerater temperature--say 3°C. It is placed in the kitchen at--say--25°C. If $\overline{h}$ = 13.5 W/m^2-°C, how long will it take for the beverage to reach 15°C? (This would be a 12 oz. can.)

Additional assumptions: The beverage has the properties of water. The aluminum offers no thermal capcitance. The bottom of the can resting on the table is insulated.

The beverage along the inner walls of the can will warm up first and rise, then stir in with the cooler liquid toward the middle. That action, not conduction, will keep the temperature close to a mean value as it warms. The Biot number is thus meaningless. But the liquid will nevertheless remain close to a constant temperature and the lumped capacitance assumption will be valid.

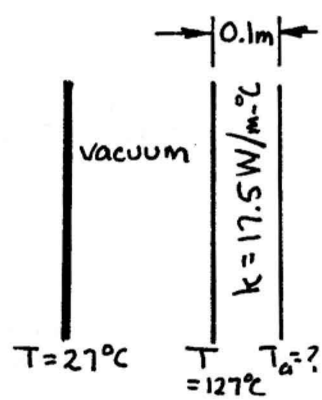
$$T = \frac{\rho c V}{\overline{h}A} = \frac{1000(4190)\pi(0.03)^2(0.12)}{13.5[\pi(.03)^2+\pi(.06)(0.12)]} = 4138 \text{ sec.}$$

We want $\dfrac{T-T_\infty}{T_i-T_\infty} = \dfrac{15-25}{3-25} = 0.4546$, so eqn. (1.21) gives:

$$0.4546 = e^{-t/4138}$$

Thus the cooling time, t, = $\underline{\underline{3263 \text{ sec}}}$ or $\underline{\underline{54.4 \text{ min}}}$

1.25 Find the far temperature of the 0.1 m wall, desinated as T_a. Assume that both walls are black



$$-k\frac{\Delta T}{L} = \sigma([127+273]^4 - [27+273]^4)$$

$$17.5\,\frac{T_a-127}{0.1} = 5.67(4^4-3^4) = 992.25$$

so $\underline{\underline{T_a = 132.7°C}}$

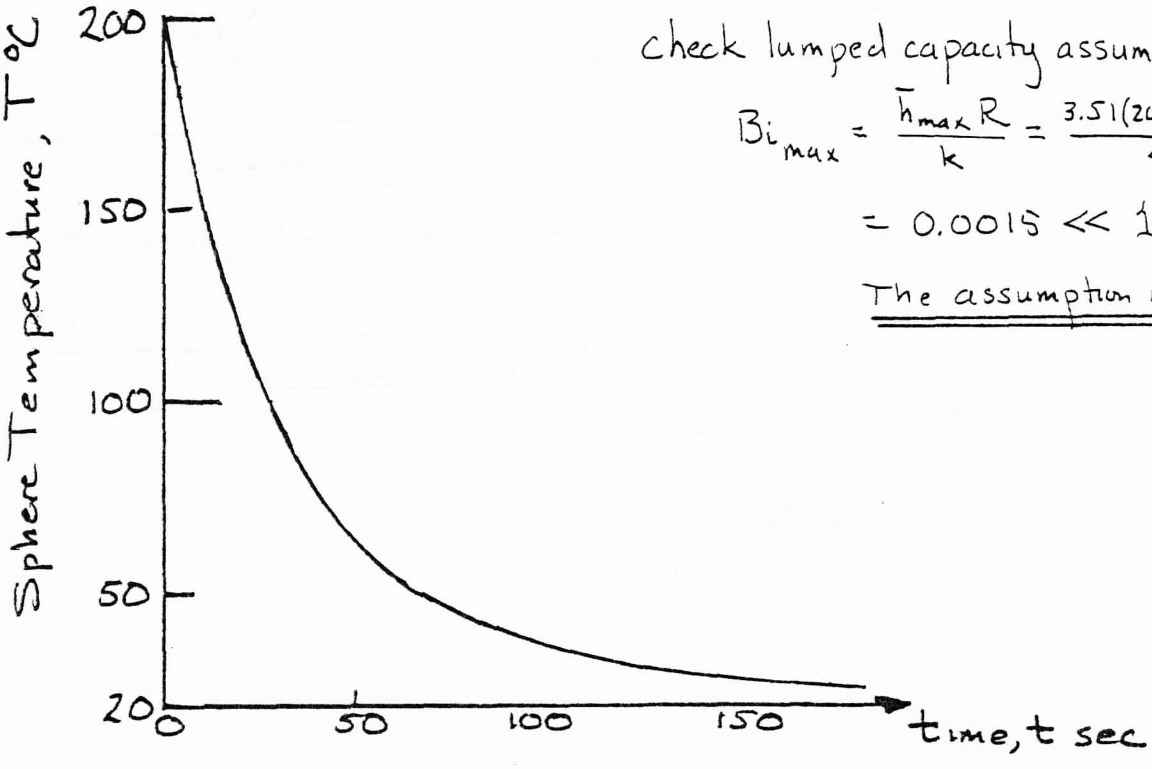
1.26 A 1 cm diam, 1% carbon steel, sphere, initially at 200°C, is cooled by natural convection with air at 20°C. In this case h is not independent of temperature. Instead, $\bar{h} = 3.51(\Delta T\,°C)^{1/4}$ W/m²-°C. Plot T_{sphere} as a function of t. Verify the lumped capacity assumption.

$$\frac{d(T-T_\infty)}{dt} = -\frac{\bar{h}A}{\rho c V}(T-T_\infty) = -\frac{3.51}{\rho c(R/3)}(T-T_\infty)^{5/4}$$

or
$$\int_{T=200}^{T}\frac{d(T-T_\infty)}{(T-T_\infty)^{5/4}} = -\int_0^t \frac{10.53\,dt}{\rho c R} = -\frac{10.53}{\rho c R}\,t$$

or
$$-\frac{1}{4}\left[\frac{1}{(T-20)^{1/4}} - \frac{1}{180^{1/4}}\right] = -\frac{10.53}{7801(473)(0.005)}\,t = -0.000571\,t$$

so
$$t = 438\left[\frac{1}{(T-20)^{1/4}} - 0.273\right]$$



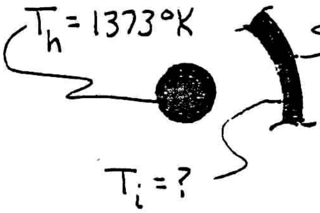
Check lumped capacity assumption

$$Bi_{max} = \frac{\bar{h}_{max}R}{k} = \frac{3.51(200-20)^{1/4}(0.005)}{43}$$

$$= 0.0015 \ll 1$$

The assumption is OK.

1.27 A 3 cm diam., black spherical heater is kept at 1100°C. It radiates, through an evacuated annulus, to a surrounding spherical shell of Nichrome V. The shell has a 9 cm ID and is 0.3 cm thick. It is black on the inside and it is held at 25°C on the outside. Find: a) The temperature of the inner wall of the shell and b) the heat transfer, Q. (Treat the shell as a plane wall.)



$$Q_{rad} = Q_{cond.}$$

$$\sigma A_h (T_h^4 - T_i^4) = k \frac{T_i - T_o}{\text{thickness}} A_s$$

divide by $A_h = \pi D_h^2$ and get $\quad 5.67(10)^{-8}(1373^4 - T_i^4) = 10\,\frac{T_i - 298.15}{0.003}\underbrace{\left(\frac{D_s}{D_h}\right)^2}_{=9}$

we solve this for T_i, by trial and error. The result is:

$$T_i = 304.85\ K = \underline{\underline{32.7°C}}$$

Then:

$$Q = Q_{rad} = 5.67\times10^{-8}\left[\pi(0.03)^2\right](1373^4 - 304.85^4)$$

$$= \underline{\underline{568\ W}}$$

Checking: $Q = Q_{cond} = 10\,\frac{(304.85 - 298.15)}{0.003}\left[\pi(0.09)^2\right]$

$$= \underline{\underline{568\ W}}$$

Problem 1.28 The sun radiates 650 W/m^2 on the surface of a particular lake. At what rate, $\dot{m}$ mm/hr, would the lake evaporate away if all this energy went to evaporating water? Discuss as many other ways this energy can be distributed as you can think of. (h_{fg} for water is 2,257,000 J/kg.) Do you suppose much of the incident radiation goes to evaporation?

$$Q_{rad} = Q_{latent}$$

$$650 \text{ J/m}^2\text{-s} = \rho_f \text{ kg/m}^3 \, [\, \dot{m} \,/1000(3600) \text{ m/s}] \, h_{fg} \text{ J/kg}$$

$$= [997(2{,}257{,}000)/3{,}600{,}000] \, \dot{m}$$

So the maximum possible $\dot{m}$ would be 1.04 mm/hr

Other places Q_{rad} could go, include:

(a) Much of the visible portion of Q_{rad} reflects off the surface. (The visible part of solar radiation is quite large so this could be important. See Fig. 11.2.)

(b) Some of the visible and ultra-violet parts of Q_{rad} will be transmitted and/or absorbed below the surface.

(c) The infrared portion, which is absorbed at the surface, can go three ways:
- Natural convection will return some to the air.
- Some will conduct to the colder water below.
- Some will reradiate to the sky at night.

Thus, the surface cannot get very hot, and the rate of evaporation should be *far* less than our calculated value of 1.04 mm/hr.

1.29 It is proposed to make picnic cups, 0.005 m thick, of a new plastic for which $k = k_o(1+aT^2)$ where T is expressed in °C, $k_o = 0.15$ W/m-°C, and $a = 10^{-4}$ °C^{-2}. We are concerned with thermal behavior in the extreme case in which T=100°C in the cup and 0°C outside. Plot T against position in the cup wall and find the heat loss, q.

In this case:

$$q = k_o(1+aT^2)\frac{dT}{dx} = \text{constant}$$

so

$$\frac{q}{k_o}\int_0^{L=0.005} dx = \int_{T(x=0)}^{T(x=L)} (1+aT^2)\,dT$$

where $T(x=L)=100$, $T(x=0)=0$

or

$$\frac{qL}{k_o} = \left[T + \frac{a}{3}T^3\right]_0^{100}$$

so:

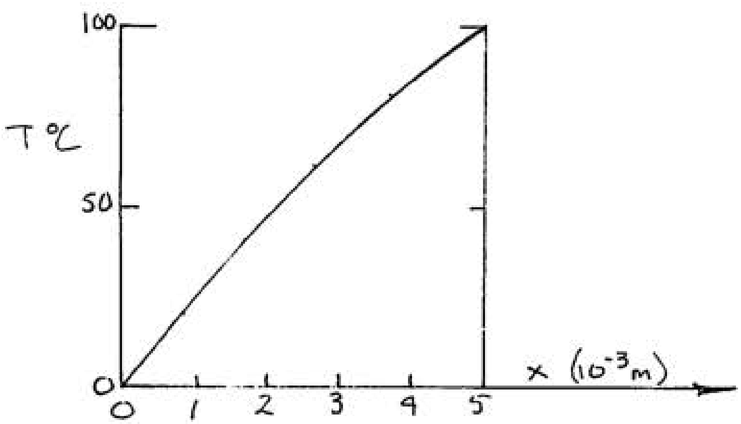
$$q = \frac{0.15}{0.005}\left[100 + \frac{10^{-4}}{3}10^6\right] = \underline{\underline{4000\ \text{W/m}^2}}$$

Had we integrated only to x of interest -- not to L, then:

$$q = 4000 = \frac{k_o}{x}\left[T(x) + \frac{a}{3}T^3(x)\right]$$

or

$$\underline{\underline{x = \frac{0.15}{4000}\left[T(x) + \frac{10^{-4}}{3}T^3(x)\right]}}$$



(Note: Chose a value of T and calculate the corresponding x. Don't get hung up solving for T at given x's, by trial & error.)

1.30

A disc-shaped wafer of diamond IIb is the target of a very high intensity laser. The disc is 5 mm in diam. and 1 mm deep. The flat side is pulsed intermittently with 10^{10} W/m^2 of energy for one microsecond. It is then cooled by natural convection from that same side until the next pulse. If $\bar{h}$=10 W/m^2-°C and T_∞=30°C, plot T_{disc} as a function of time for pulses that are 50 sec apart and 100 sec apart. (Note that you must determine the temperature the disc reaches before it is pulsed, each time.)

First evaluate $\dfrac{1}{T} = \dfrac{\bar{h}A}{\rho c V} = \dfrac{10\,\frac{\pi}{4}(0.005)^2}{3250(510)\frac{\pi}{4}(0.005)^2(0.001)} = \dfrac{1.963\,(10)^{-4}}{0.03254} = \dfrac{1}{165.8}$

The disc receives $10^{10}\,\dfrac{J}{m^2\text{-}s} \times \dfrac{\pi}{4}(0.005)^2\,m^2 \times 10^{-6}\,s = 0.1964\,\dfrac{J}{\text{pulse}}$

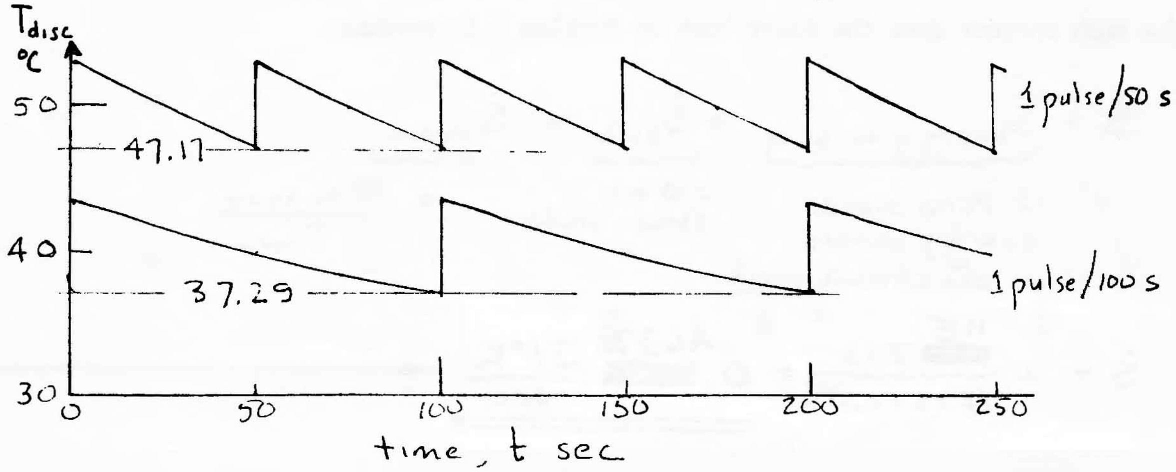
so its temperature rises:

$$T_i - T_f = \frac{0.1964\ J}{\rho c V\ J/°C} = 6.034\ \frac{°C}{\text{pulse}}$$

The cooling will be as described by eqn. (1.21)

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-t/165.8} \qquad \text{so} \qquad \frac{T_f - 30}{T_f - (30 - 6.034)} = e^{-\frac{50 \text{ or } 100}{165.8}}$$

We solve this for T_f and get $T_f = 47.17$ for 50 sec interval
Then using eqn. (1.21) we plot: $\quad = 37.29$ for 100 sec interval



1.31 An old incandescent 60W light bulb is roughly a 0.06m diam. sphere. Its surface temp. is 115°C. The average heat transfer coefficient outside the bulb is 8.2W/m°K

(a) Show that the peak radiation from the glass to the room has a *near-infrared* wavelength.

(b) What is the heat loss from the glass if $\varepsilon_{glass} = 0.94$?

(c) How much heat transfer remains to occur by direct radiation from the filament through the glass? Most of that energy is not in the visible spectrum. These bulbs were very inefficient.)

a.) Wien's Law, eqn. 1.29: μ at max $e_b = 2898(115 + 273)$

$$= 7.47\ \mu m$$

This places the radiation: squarely in the near-infrared

(See Table 1.2)

b.) Q from glass $= \overline{h}A\Delta T + \sigma A(T^4_{bulb} - T^4_{\infty})\ \varepsilon_{glass}$

$$= 7\pi(0.06)^2(115-25) + 5.67(10)^8\pi(388^4 - 290^4)0.34$$

$$= 7.125\ W + 9.397\ W = \underline{16.52\ W}$$

c.) Percent of direct radiation from filament

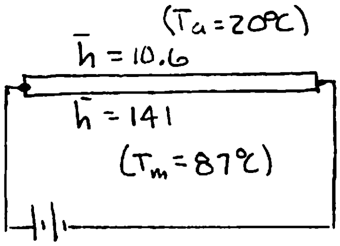
$$= (115 - 16.52)100/115 = \underline{85.6\text{ percent}}$$

1.32 How much entropy does the light bulb in Problem 1.31 produce?

$$\dot{S} = \underbrace{\dot{S}_{\text{energy to bulb}}}_{\substack{0 \text{ since energy}\\ \text{enters as}\\ \text{electrical work}}} + \underbrace{\dot{S}_{\text{bulb}}}_{\substack{0 \text{ at steady}\\ \text{state}}} + \underbrace{\dot{S}_{\text{room}}}_{= \frac{Q_{\text{to room}}}{T_{\text{room}}}}$$

$$= 0 + 0 + \frac{115\ J/s}{(273+25)K} = \underline{\underline{0.4637\,\frac{J}{K\,sec}}}$$

1.33 Air at 20°C flows over one side of a thin metal sheet ($\bar{h}$=10.6W/m^2-°C). Methanol at 87°C flows over the other side ($\bar{h}$=141 W/m^2-°C). The metal functions as an electrical resistance heater, releasing 1000 W/m^2. Calculate: a) the heater temperature, b) the heat transfer from the methanol to the heater, and c) the heat transfer from the heater to the air.



$$q_{\text{to plate from meth.}} + q_{\text{elect. to plate}} = q_{\text{to air from plate}}$$

$$141(87 - T_{plate}) + 1000 = 10.6(T_{plate} - 20)$$

$$\underline{\underline{T_{plate} = 88.91\,°C}}$$

Then $q_{\text{meth to heater}} = 141(87 - 88.91) = \underline{\underline{-269.5\ W/m^2}}$

$q_{\text{heater to air}} = 10.6(88.91 - 20) = \underline{\underline{730.5\ W/m^2}}$

1.34 A black heater is simultaneously cooled by 20°C air (with $\bar{h}$=14.6 W/m^2-°C) and by radiation to a parallel black wall at 80°C. What is the temperature of the first wall if it delivers 9000 W/m^2.

$$9000 = 14.6(T_w - [20+273]) + 5.67(10)^{-8}(T_w^4 - [80+273]^4)$$

or

$$T_w^4 = 24.97(10)^{10} - 0.02575(10)^{10}\, T_w$$

Solve by trial & error:

trial T_w	LHS	RHS
600°K	$12.96(10)^{10}$	$9.52(10)^{10}$
550	9.15	10.81
560	9.835	10.55
565	10.19	10.42
568	10.41	10.34
567	10.34	10.37

We obtain

$T_{wall} = 567\ °K$

or $\underline{\underline{= 294°C}}$

(this gives 44.4% by convection and 55.6% by radiation.)

Problem 1.35: A 250 mL (8.3 oz.) aluminum beverage can is taken from a 3°C refrigerator and placed in a low humidity, 25°C room ($\overline{h}$ = 7.3 W/m²K). The 53.3 mm diameter by 112 mm high can is placed on an insulated surface. How long will it take to reach 12°C? Assume that emittance of the can is very low, so thermal radiation is negligible. Discuss your other approximations.

Solution: The beverage has about the same properties as water (ρ = 999.9 kg/m³, c = 4200 J/kg K). We can neglect the heat capacitance of the thin aluminum can. We can also assume that the liquid in the can will circulate under buoyancy, so internal temperature gradients will remain small. Then a lumped capacitance solution applies.

The surface area for heat transfer is the top and sides of the can:

$$A = \pi D^2/4 + \pi DL = \pi\,[(53.3)^2/4 + (53.3)(112)] \times 10^{-6}\ \text{m}^2 = 0.02099\ \text{m}^2$$

and the mass of liquid is:

$$m = \rho V = (999.9\ \text{kg/m}^3)(250 \times 10^{-6}\ \text{m}^3) = 0.2500\ \text{kg}$$

The time constant is

$$\boldsymbol{T} = \frac{mc}{\overline{h}A} = \frac{(0.2500)(4200)}{(7.3)(0.02099)}$$

$$= 6953\ \text{s}$$

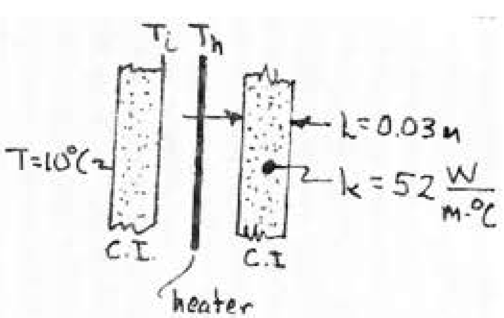
Using eqn. (1.22) with T_i = 3°C and T_∞ = 25°C, we solve for the time at which T = 12°C:

$$\frac{12 - 25}{3 - 25} = e^{-t/6853}$$

$$0.5909 = e^{-t/6853}$$

$$t = -\ln(0.5909)\cdot(6853\ \text{s}) = 3605\ \text{s} = \underline{1\ \text{h}\ 5\ \text{s}}$$

Problem 1.36: A thin sheet is a resistance heater, parallel with 3 cm slabs of cast iron on either side, in an evacuated cavity. The heater, which releases 8000 W/m², and the cast iron, are very nearly black. The outside surfaces of the cast iron slabs are held at 10°C. Find the temperatures of the heater and the inside of the slabs.



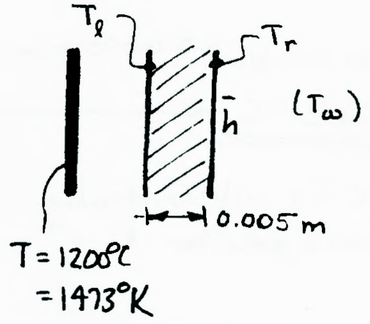
$q = 8000/2 = k\,(T_i - 10)/L = 52(T_i - 10)/0.03$

so: $T_i = \underline{12.3°C = 285.5\ K}$

Then: $q = 4000 = \sigma(T_h^4 - T_i^4)$ so

$T_h = [4000/5.67(10)^{-8} + 285.5^4]^{1/4} = \underline{527\ K = 254°C}$

1.37 A black wall at 1200°C radiates to the left side of a parallel slab of 316 stainless steel, 5 mm thick. The right side of the slab is to be cooled convectively and is not to exceed 0°C. Suggest a convective process that will achieve this.



$$\sigma(1473^4 - T_\ell^4) = \underbrace{k_{ss}}\,\frac{T_\ell - 273}{0.005}$$

use 16 W/m-°C at $\bar{T}$ = 200°C

$$T_\ell^4 = -5.644(10)^{10}\,T_\ell - 2.012(10)^{13}$$

By Trial & error, get $T_\ell = 357°K$

Thus we can calculate q when T_r is at its highest value:

$$q = k_{ss}\frac{T_\ell - 273}{0.005} = 3200(357-273) = \underline{268,800\ W/m^2}$$

Consider some $\bar{h}$'s: if $T_\infty = -10°C$, $\bar{h} = 26,800$
" " = −50°C, h = 5360
" " = −100°C, h = 2680

Looking at Table 1.1 we see natural convection won't work. The temperature is too low to involve water. Forced convection will probably be the practical means. Perhaps a low freezing point organic liquid (at very high velocity), but more likely a cryogenic liquid boiling as it flows by.

1.38 A cooler keeps one side of a 2 cm layer of ice at -10°C. The other side is exposed to air at 15°C. What is $\bar{h}$, just on the edge of melting? Must $\bar{h}$ be raised or lowered if melting is to progress?

$$k_{ice}\frac{T_{outer} - (-10)}{\text{thickness}} = \bar{h}(T_\infty - T_{outer}),$$ where $T_{outer} = 0°C$ at melting

So:

$$2.215\,\frac{0+10}{0.02} = \bar{h}\cdot(15-0), \quad \bar{h}_{\text{incipient melting}} = \underline{\underline{73.8\,W/m^2\text{-}°C}}$$

if $T_{outer} < 0°C$, $\bar{h}$ will be lower. Therefore $\bar{h}$ must to raised to make melting progress.

Problem 1.39 At what minimum temperature does a black radiator have its maximum monochromatic emissive power in the visible wavelength range? Look at Fig. 10.2; then describe the difference between what you might see looking at this object in comparison to looking at the sun.

Solution In accordance with eqn. (1.28), and using $\lambda_{max.visible} = 0.00008$ cm,

$$0.00008 \text{ T} = 0.2898 \text{ cm K}. \quad \text{Therefore } \underline{\underline{T = 3623 \text{ K}}}$$

The sun radiates at about 5777 K (see Fig. 10.2). This is a substantially higher temperature. It also delivers its maximum e_λ at a much smaller wavelength – one at the lower end of the visible range.

Problem 1.40 The local heat transfer coefficient during the laminar flow of fluid over a flat plate of length L is equal to $F/x^{1/2}$, where F is some function of fluid properties and the flow velocity. How does the average heat transfer coefficient compare with h(x=L) if x is the distance from the leading edge of the plate?

Solution: We use the definition of the average to get:

$$\bar{h} = \frac{1}{L}\int_0^L h\,dx = 2\frac{F}{L}\,x^{1/2}\Big|_0^L = 2\,\frac{F}{\sqrt{L}} = 2\,h(x = L)$$

Therefore, the average heat transfer coefficient = $\underline{\bar{h} = 2\,h(x = L)}$

Problem 1.41 An object is initially at a temperature higher than its surroundings. We have seen that many kinds of convective processes will bring the object into equilibrium with its surroundings. Describe the characteristics of a process that will do so with the smallest net increase in the entropy of the universe.

Solution Entropy is *not* a path function. <u>*Any* process connecting the initial to the final states will yield the same increase of entropy.</u>

1.42 A 250°C cylindrical copper billet, 4 cm in diameter and 8 cm long, is cooled in air at 25°C. The heat transfer coefficient is 5 W/m^2-°C. Can this be treated as lumped capacitance cooling? What is the temperature of the billet after 10 minutes?

Volume $= \frac{\pi}{4}(0.04)^2(0.08) = \underline{0.0001005\ m^3}$, Area $= 2\frac{\pi}{4}(0.04)^2 + \pi(0.04)(0.08)$
$= \underline{0.01257\ m^2}$

Base Bi on V/A = 0.008 m (We could also have used R/2 = 0.010 or some other possibilities on the same order of magnitude.)

Then $Bi = \frac{\bar{h}\,V/A}{k} = \frac{5(0.008)}{391} = 0.000102 \ll 1$. It is OK to use lumped capacity.

$$\mathbf{T} = \frac{\rho c V}{hA} = 384(8954)\frac{0.008}{5} = \underline{5501\ sec}$$

so: $\frac{T(10\ min) - 25}{250 - 25} = e^{-600\ sec/5501\ sec}$ $\qquad T(10\ min) = \underline{\underline{226.8°C}}$

PROBLEM 1.43: The diameter of the sun is roughly 1,391,000 km, and it emits energy as if it were a black body at about 5772 K. Determine the rate at which it emits energy. Compare this with the known value. How much energy does the sun emit per year? [1.21×10^{34} J/y]

Older versions of AHTT used 5777 K for the solar temperature and 1,387,000 km for the solar diameter. The results are nearly identical.

SOLUTION

The radiative power *emitted by* the sun is

$$\begin{aligned} Q_{\text{sun}} &= \pi D_{\text{sun}}^2 \sigma T_{\text{sun}}^4 \\ &= \pi(1.391 \times 10^9)^2(5.670374 \times 10^{-8})(5772)^4 \\ &= \underline{3.826 \times 10^{26}\ \text{W}} \end{aligned}$$

With the SI prefixes in Table B.1, we could instead write $Q_{\text{sun}} = 382.6$ YW ("yottawatts"). This value is a nearly exact match to the 2015 standard value of 3.828×10^{26} W [See comment below].

The annual energy is

$$E_{\text{sun}} = (365.25)(24)(3600)(3.826 \times 10^{26}) = 1.207 \times 10^{34}\ \text{J/y} = \underline{1.207 \times 10^3\ \text{QJ/y}}$$

Comment. The International Astronomical Union provides the solar data used here: Prša et al., "Nominal Values for Selected Solar and Planetary Quantities: IAU 2015 Resolution B3," *Astronomical Journal* **152**:41, 2016, doi:10.3847/0004-6256/152/2/41.

PROBLEM 1.44: Room temperature objects at 300 K and the sun at 5772 K each radiate thermal energy; but Planck's law, eqn. (1.30), shows that the wavelengths of importance are quite different.

a) Find $\lambda_{\max}$ in micrometers for each of these temperatures from Wien's Law, eqn. (1.29).
b) Using a spreadsheet or other software, plot eqn. (1.30) for $T = 300$ K as a function of wavelength from 0 to 50 μm and for $T = 5772$ K for wavelengths from 0 to 5 μm.
c) By numerical integration, find the total area under each of these curves and compare the value to the Stefan-Boltzmann law, eqn. (1.28). Explain any differences.
d) Show that about 1/4 of the area under each curve is to the left of $\lambda_{\max}$ (in other words, 3/4 of the energy radiated is on wavelengths greater than $\lambda_{\max}$).
e) What fraction of the energy radiated by the 300 K surface is carried on wavelengths less than 4 μm? What fraction of the energy radiated by the 5772 K surface is on wavelengths greater than 4 μm?

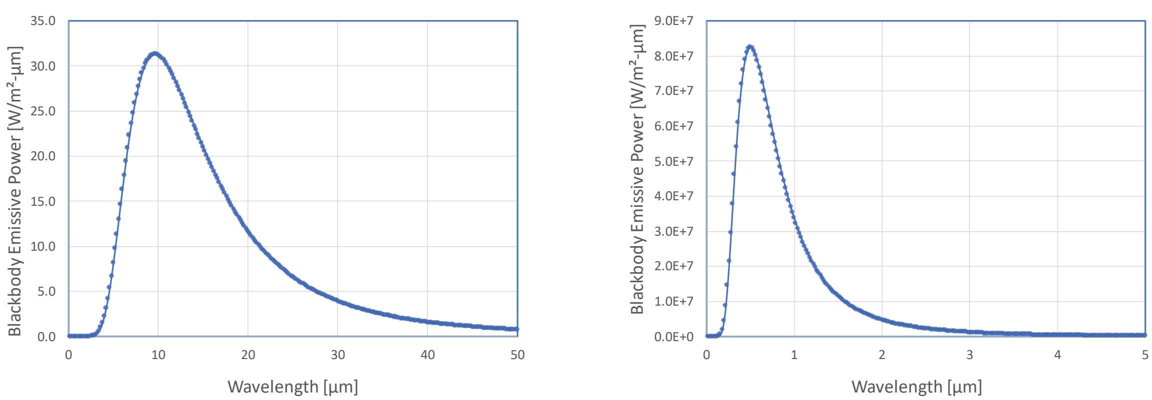
Earlier versions of AHTT used 5777 K for the solar temperature. The results are nearly identical.

SOLUTION.

a)

$$\lambda_{\max} = \frac{2987.77 \text{ μm} \cdot \text{K}}{T \text{ K}} = \begin{cases} 9.9592 \text{ μm} & \text{at 300 K} \\ 0.5176 \text{ μm} & \text{at 5772 K} \end{cases}$$

b) The plotting and integration can be done in various ways depending upon what software is used. The results in Fig. 1 are from an Excel spreadsheet with a step size of 0.2 μm at 300 K and of 0.02 μm at 5772 K.



(A) 300 K (B) 5772 K

FIGURE 1. Plots of Planck's law at two temperatures

c) Using the values from Excel, a trapezoidal rule integration gives the area under the curve:

$$\text{Integrated area} = \begin{cases} 445.2 \text{ W/m}^2 & \text{at 300 K} \\ 6.261 \times 10^7 \text{ W/m}^2 & \text{at 5772 K} \end{cases}$$

The Stefan-Boltzmann law yields (with $\sigma = 5.670374 \times 10^{-8}$ W/m^2 K^4)

$$\sigma T^4 = \begin{cases} 459.3 \text{ W/m}^2 & \text{at } 300 \text{ K} \\ 6.294 \times 10^7 \text{ W/m}^2 & \text{at } 5772 \text{ K} \end{cases}$$

At 5772 K, the integrated value is 99.48% of the Stefan-Boltzmann law, and at 300 K, it is 96.93%. The principal reason that these values are low is that energy is also radiated at wavelengths higher than the range of integration.

d) By integrating up to $\lambda_{\max}$ from part a),

$$\frac{1}{\sigma T^4} \int_0^{\lambda_{\max}} e_{\lambda,b}(T)\, d\lambda = \begin{cases} 28.4\% & \text{at } 300 \text{ K} \\ 29.6\% & \text{at } 5772 \text{ K} \end{cases}$$

These values are bit more than 1/4 of the total energy, but are often stated as “about 1/4”.

e) Similar integrations show that a 300 K surface radiates only 0.33% on wavelengths below 4 μm and that a 5772 K surface radiates 99.0% on wavelengths less than 4 μm (or 1% on wavelengths above 4 μm). This fact enables the design of materials that selectively absorb or reflect solar energy (see Section 10.6).

Problem 1.45

A crucible of molten metal at 1800°C is placed on the foundry floor. The foundryman covers it with a metal sheet to reduce heat loss to the room. If $\mathscr{F}$ is 0.4 between the melt and the plate and 0.8 between either the melt or the top of the plate and the room, how much will the heat loss to the room be reduced by the sheet?

Solution. First find the sheet temperature:

$$q = (0.8)\sigma\left[T_s^4 - (20 + 273)^4\right] = (0.4)\sigma\left[(1800 + 273)^4 - T_s^4\right]$$

This gives $T_{\text{sheet}} = 1575$°K, so

$$\frac{q_{\text{with sheet}}}{q_{\text{without sheet}}} = \frac{0.8\sigma(1575^4 - 293^4)}{0.8\sigma(2073^4 - 293^4)} = 0.333$$

The heat loss is therefore reduced by 66.7% by the shield.

Problem 1.46: Integration of Planck's law, eqn. (1.30) over all wavelengths leads to the Stefan-Boltzmann law, eqn. (1.28). Perform this integration and determine the Stefan-Boltzmann constant in terms of other fundamental physical constants. *Hint:* The integral can be written in terms of Riemann's zeta function, $\zeta(s)$, by using this beautiful relationship between the zeta and gamma functions

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt$$

for $s > 1$. When s a positive integer, $\Gamma(s) = (s-1)!$ is just a factorial. Further, several values of $\zeta(s)$ are known in terms of powers of π and can be looked up.

Solution.

$$e_b(T) = \int_0^\infty e_{\lambda,b}\,d\lambda \tag{1}$$

$$= \int_0^\infty \frac{2\pi h c_o^2}{\lambda^5\left[\exp(hc_o/k_B T\lambda) - 1\right]}\,d\lambda \tag{2}$$

$$= \int_0^\infty \frac{2\pi h \nu^3}{c_o^2\left[\exp(h\nu/k_B T) - 1\right]}\,d\nu \tag{3}$$

$$= \frac{2\pi k_B^4 T^4}{h^3 c_o^2}\int_0^\infty \frac{x^3}{e^x - 1}\,dx \tag{4}$$

We are given

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt$$

For our case, $s = 4$ and $\Gamma(4) = 3! = 6$. Hence:

$$e_b(T) = \frac{2\pi k_B^4 T^4}{h^3 c_o^2}\,\zeta(4)\,3! \tag{5}$$

$$= \frac{12\pi k_B^4}{h^3 c_o^2}\zeta(4)\,T^4 \tag{6}$$

Zeta is a famous function, and the value at 4 has been established to be:

$$\zeta(4) = \frac{\pi^4}{90}$$

Hence:

$$e_b(T) = \left(\frac{2\pi^5 k_B^4}{15 h^3 c_o^2}\right) T^4 \tag{7}$$

$$= \sigma\,T^4 \tag{8}$$

where we have also found the Stefan-Boltzmann constant in terms of fundamental physical constants.